TOPOLOGY OF $U(2,1)$ REPRESENTATION SPACES

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Abstract. The Betti numbers of moduli spaces of representations of a universal central extension of a surface group in the groups $U(2,1)$ and $SU(2,1)$ are calculated. The results are obtained using the identification of these moduli spaces with moduli spaces of Higgs bundles, and Morse theory, following Hitchin's programme [14]. This requires a careful analysis of critical submanifolds which turn out to have a description using either symmetric products of the surface or moduli spaces of Bradlow pairs.

1. Introduction

Let X be a closed oriented surface of genus $g \geqslant 2$. Consider the universal central extension

$$0 \to \mathbb{Z} \to \Gamma \to \pi_1 X \to 0,$$

generated by the standard generators $a_1, b_1, \dots, a_g, b_g$ of $\pi_1 X$ and a central element J with the relation $J = \prod_{i=1}^{g} [a_i, b_i]$. Our object of study is the moduli space of reductive representations

$$\mathcal{M}_{U(2,1)} = \mathrm{Hom}^{+}(\Gamma, U(2,1))/U(2,1)$$

of Γ in the non-compact Lie group $U(2,1)$. As is well known such representations correspond to flat $U(2,1)$-bundles on the punctured surface $X \setminus \{p\}$ with fixed holonomy around the puncture given by the image of the central element J in $U(2,1)$ under the representation. Such bundles extend (as non-flat bundles) over the puncture and they are topologically classified by their reduction to the maximal compact subgroup $U(2) \times U(1) \subseteq U(2,1)$, that is, by a pair of integers (d_1, d_2), where d_1 is the degree of the rank 2 complex vector bundle given by projecting $U(2) \times U(1) \to U(2)$ and d_2 is the degree of the complex line bundle given by projecting onto $U(1)$. These characteristic numbers are subject to the bound

$$|d_1 - 2d_2| \leqslant 3g - 3; \tag{1.1}$$

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this follows from the work of Domic and Toledo [7] and can also be proved using Higgs bundles (see Xia [21]). Furthermore, Xia proved that the subspaces $\mathcal{M}_{d_1,d_2}$ of representations with characteristic numbers (d_1, d_2) are exactly the connected components of $\mathcal{M}_{U(2,1)}$.

In this paper we calculate the Betti numbers of the spaces $\mathcal{M}_{d_1,d_2}$ in the case when $(d_1 + d_2, 3) = 1$. (We need to impose this condition in order to make sure that the moduli space is non-singular.) We use the approach via Higgs bundles and Morse theory introduced by Hitchin [14]: $\mathcal{M}_{d_1,d_2}$ is homeomorphic to the moduli space of solutions to a set of equations from gauge theory known as Hitchin's equations and this space can be identified with an algebro-geometric moduli space of Higgs bundles of a certain special form. The point of view of gauge theory allows one to do Morse theory in the sense of Bott on the moduli space and the point of view of algebraic geometry permits a fairly explicit description of the critical submanifolds in terms of known spaces: the critical submanifolds turn out to be either closely related to symmetric products of the surface or, more interestingly, to moduli spaces of Bradlow pairs.

The formula for the Betti numbers of $\mathcal{M}_{U(2,1)}$ is given in Theorem 3.3 and is fairly complicated. Of course one can obtain more explicit results in low genus: see (3.10) and (3.11) for the Poincaré polynomials of the two connected components of $\mathcal{M}_{U(2,1)}$ in the case $g = 2$ and $d = 1$.

A minor modification of our calculations gives the Betti numbers of the closely related moduli space $\mathcal{M}_{SU(2,1)}$ of reductive representations of Γ in $SU(2,1)$ (Theorem 4.1). This space has an interpretation as a moduli space of fixed determinant Higgs bundles and $\mathcal{M}_{U(2,1)}$ fibres over the Jacobian of X with fibres isomorphic to $\mathcal{M}_{SU(2,1)}$. By analogy to the case of the moduli space of stable vector bundles one might expect the Poincaré polynomial of the non-fixed determinant moduli space to be the product of that of the fixed determinant moduli space by that of the Jacobian (see Atiyah and Bott [1] and Harder and Narasimhan [12]). It is noteworthy that this is not the case in our situation and, in particular, it follows that the group of 3-torsion points in the Jacobian of X acts non-trivially on the rational cohomology of $\mathcal{M}_{SU(2,1)}$ (Proposition 4.2).

Another interesting aspect is that the Euler characteristic of the moduli spaces can be calculated. The components of $\mathcal{M}_{U(2,1)}$ all have zero Euler characteristic—this of course already follows from the fact that they fibre over the Jacobian which itself has zero Euler characteristic. More interestingly, the components of $\mathcal{M}_{SU(2,1)}$ have non-zero Euler characteristic (see (4.4)). Again this is in contrast to the case of the moduli space of stable bundles of fixed determinant which has zero Euler characteristic.

This paper is organized as follows: in Section 2 we recall the necessary background on Higgs bundles and the Morse theory strategy; in Section 3 we analyze the critical submanifolds and determine their Betti numbers; finally, in Section 4, we treat the fixed determinant moduli spaces.

2. Higgs bundles and Morse theory on the moduli space

In this section we outline the strategy of our calculations and recall the necessary background. For details on this material see the papers of Corlette [5], Donaldson [8], Hitchin [14, 15], and Simpson [18, 19].

Give X the structure of a Riemann surface. The space $\mathcal{M}_{U(2,1)}$ is homeomorphic to the moduli space of poly-stable Higgs bundles (E,ϕ) of the form

$$\begin{aligned} E &= E_1 \oplus E_2 \\ \phi &= \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right), \end{aligned} \tag{2.1}$$

where E_1 is a rank 2 holomorphic bundle and E_2 is a holomorphic line bundle on X. Furthermore, the Higgs field ϕ consists of two holomorphic maps

$$b\colon E_2 \to E_1 \otimes K, \quad c\colon E_1 \to E_2 \otimes K,$$

where K is the canonical bundle of X. A Higgs bundle (E,ϕ) is called stable if the usual slope stability condition $\mu(F) < \mu(E)$ is satisfied for any proper non-zero ϕ-invariant subbundle $F \subseteq E$ (recall that the slope of a holomorphic bundle E is $\mu(E) = \deg(E)/\operatorname{rk}(E)$). In fact it is sufficient to consider subbundles of the form $F = F_1 \oplus E_2$, where $F_i \subseteq E_i$, $i = 1, 2$ (cf. [10]). A Higgs bundle (E,ϕ) is said to be poly-stable if it is the direct sum of stable Higgs bundles, all of the same slope.

It will be convenient to express the bound (1.1) in terms of d_2 and $d = d_1 + d_2$:

$$d/3 - (g-1) \leqslant d_2 \leqslant d/3 + (g-1); \tag{2.2}$$

for fixed d there is thus one connected component $\mathcal{M}_{d_1,d_2}$ for each value of d_2 in this range.

Note that, clearly, $d_1 = \deg(E_1)$ and $d_2 = \deg(E_2)$. For purposes of topology we can therefore identify $\mathcal{M}_{d_1,d_2}$ with the moduli space of poly-stable Higgs bundles of the form (2.1) with $\deg(E_i) = d_i$, $i = 1, 2$. Note also that taking a Higgs bundle of the form (2.1) to its dual $(E_1^* \oplus E_2^*, \phi^t)$ gives an isomorphism of the corresponding components of the moduli space. We may therefore assume that $\mu(E_1) \leqslant \mu(E_2)$ or, equivalently, that $3d_2 - d \geqslant 0$. This, together with (2.2), gives the range

$$d/3 \leqslant d_2 \leqslant d/3 + g - 1 \tag{2.3}$$

for d_2.

If $\deg(E) = d = d_1 + d_2$ is co-prime to $3 = \operatorname{rk}(E)$ there are no strictly poly-stable Higgs bundles and in this case the moduli space is smooth. This is essential to doing Morse theory on it so we shall make this assumption from now on.

Considering the moduli space from the point of view of gauge theory allows one to have metrics on the Riemann surface and the bundles E_1 and E_2. It therefore makes sense to consider the function

$$f = \|\phi\|^2$$

on the moduli space. This function is a perfect Morse-Bott function and so can be used to calculate the Poincaré polynomial of the moduli space:

$$\begin{aligned} P_t(\mathcal{M}_{U(2,1)}) &= \sum_i \dim(H^i(\mathcal{M}_{U(2,1)}; \mathbb{Q}))t^i \\ &= \sum_N t^{\lambda_N} P_t(N) \end{aligned} \tag{2.4}$$

where the sum is over the critical submanifolds N of f, and the index λ_N is the real dimension of the subbundle of the normal bundle of N on which the Hessian of f is negative definite.

In order to carry out the calculation it is therefore necessary to be able do determine the critical submanifolds of f and their indices: a Higgs bundle (E, ϕ) is a critical point of f if and only if it is a variation of Hodge structure, i.e., it is of the form

$$E = F_1 \oplus \cdots \oplus F_m,$$

where the Higgs field ϕ maps F_i to $F_{i+1} \otimes K$. Furthermore, in our case each F_i must be a subbundle of E_1 or E_2. The Morse indices can be calculated in terms of the invariants of the bundles F_i (see Section 2.5 of [10]): setting $U_k = \bigoplus_{k=i-j} \operatorname{Hom}(F_j, F_i)$, the Morse index at the critical point corresponding to (E, ϕ) is

$$\lambda = 2 \sum_{k=2}^{m-1} \big((g-1) \operatorname{rk}(U_k) + (-1)^{k+1} \deg(U_k) \big). \tag{2.5}$$

In a similar manner the complex dimension of the critical submanifold containing (E, ϕ) is

$$1 + (g-1)\big(\operatorname{rk}(U_1) + \operatorname{rk}(U_0)\big) + \deg(U_1) - \deg(U_0) \tag{2.6}$$

and so the complex dimension of the downwards Morse flow of the critical submanifold through (E, ϕ) is given by

$$1 + \sum_{k=0}^{m-1} \big((g-1) \operatorname{rk}(U_k) + (-1)^{k+1} \deg(U_k) \big). \tag{2.7}$$

From this and the determination below of the critical submanifolds one can easily show that the dimension of the downwards Morse flow is not

the same for all critical submanifolds. This is in contrast to the case of moduli spaces of representations in a complex group: it was shown in [10] that in this case the dimension of the downwards Morse flow is exactly half the dimension of the moduli space reflecting two fundamental facts about the moduli space of Higgs bundles: Hausel's theorem [13] that the downwards Morse flow coincides with the nilpotent cone (the fibre over 0 of the Hitchin map) and Laumon's theorem [16] that the nilpotent cone is a Lagrangian submanifold.

3. Critical submanifolds: Bradlow pairs and symmetric products

Next we turn to the detailed analysis of the critical submanifolds. This is analogous to the analysis in [11], where the Betti numbers for the moduli space of rank 3 Higgs bundles (corresponding to representations of Γ in $SL(3,\mathbb{C})$) were calculated.

Note that a Higgs bundle of the form (2.1) with $\phi = 0$ cannot be stable since at least one of the ϕ-invariant subbundles E_1 and E_2 will violate the stability condition. It follows that a critical point is represented by a chain $E = \bigoplus_{i=1}^{m} F_i$ of length $m = 2$ or $m = 3$.

It turns out that the length 2 critical points are essentially what is known as *holomorphic triples*. These are generalizations of Bradlow pairs [2] and were introduced by García-Prada in [9]; they were later studied systematically by Bradlow and García-Prada in [4]. We briefly recall the relevant definitions: a holomorphic triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ consists of two holomorphic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ and a holomorphic map $\Phi\colon \mathcal{E}_2 \to \mathcal{E}_1$. A holomorphic sub-triple is defined in the obvious way. For $\alpha \in \mathbb{R}$ the triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is said to be α-stable if

$$\mu(\mathcal{E}_1' \oplus \mathcal{E}_2') + \alpha \frac{\operatorname{rk}(\mathcal{E}_2')}{\operatorname{rk}(\mathcal{E}_1') + \operatorname{rk}(\mathcal{E}_2')} < \mu(\mathcal{E}_1 \oplus \mathcal{E}_2) + \alpha \frac{\operatorname{rk}(\mathcal{E}_2)}{\operatorname{rk}(\mathcal{E}_1) + \operatorname{rk}(\mathcal{E}_2)}$$

for any proper non-trivial sub-triple $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$.

We have the following proposition concerning critical points represented by length 2 chains.

Proposition 3.1. *There is one critical submanifold $\mathcal{N}^2$ of $\mathcal{M}_{d_1,d_2}$ consisting of length 2 chains. This critical submanifold is isomorphic to the moduli space of α-stable holomorphic triples $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ where $\alpha = 2g-2$, $\operatorname{rk}(\mathcal{E}_1) = 2$, $\operatorname{rk}(\mathcal{E}_2) = 1$, $\deg(\mathcal{E}_1) = 4g-4+d_1$, and $\deg(\mathcal{E}_2) = d_2$. Furthermore, the Morse index of $\mathcal{N}^2$ is*

$$\lambda(\mathcal{N}^2) = 0,$$

Proof. This is analogous to Proposition 2.9 of [11], (cf. also Theorem 5.2 of [10]): under our assumptions a length 2 chain must have $F_1 = E_2$, $F_2 = E_1$ and $c = 0$; thus setting $\mathcal{E}_1 = E_1 \otimes K$, $\mathcal{E}_2 = E_2$ and $\Phi = b$ one obtains a holomorphic triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$. One then proves that the stability conditions coincide.

The Morse index is obviously zero from (2.5). □

It remains to determine the Poincaré polynomial of $\mathcal{N}^2$. As shown by García-Prada in [9], the fact that $\mathcal{E}_2$ is a line bundle implies that there is an isomorphism

$$\begin{aligned} \mathcal{N}^2 &\to \mathcal{M}^{\text{pairs}} \times \operatorname{Pic}^{d_2} X \\ (\mathcal{E}_1, \mathcal{E}_2, \Phi) &\mapsto \big((\mathcal{E}_2^* \otimes \mathcal{E}_1, \Phi), \mathcal{E}_2\big), \end{aligned}$$

where $\mathcal{M}^{\text{pairs}}$ is the moduli space of α-stable Bradlow pairs (V, Φ). Hence $P_t(\mathcal{N}^2) = (1+t)^{2g} P_t(\mathcal{M}^{\text{pairs}})$. The Poincaré polynomial of $\mathcal{M}^{\text{pairs}}$ was, essentially, determined by Thaddeus in [20]: he considered the moduli space of fixed determinant pairs, however (cf. Bradlow, Daskalopoulos and Wentworth [3]), the arguments go through in the case of non-fixed determinant pairs. The result is that $\mathcal{N}^2$ has Poincaré polynomial

$$\begin{aligned} P_t(\mathcal{N}^2) = {} & \frac{(1+t)^{4g}}{1-t^2} \\ & \cdot \operatorname*{Coeff}_{x^i} \left(\frac{t^{2\deg(V)+2g-2-4i}}{xt^4 - 1} - \frac{t^{2i+2}}{x - t^2} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right), \end{aligned} \tag{3.1}$$

where

$$i = \left[\frac{2d}{3}\right] - 2d_2 + 2g - 2$$

and $V = \mathcal{E}_2^* \otimes \mathcal{E}_1 = E_2^* \otimes E_1 \otimes K$ so that

$$\deg(V) = 4g - 4 + d - 3d_2.$$

With regard to the critical points represented by length 3 chains note that these are necessarily of the form

$$\begin{aligned} E_1 &= F_1 \oplus F_3 \\ E_2 &= F_2, \end{aligned}$$

where the F_i are line bundles and $\phi_i \colon F_i \to F_{i+1} \otimes K$. Note also that $c = \phi_1 \in H^0(F_1^{-1} F_2 K)$ and $b = \phi_2 \in H^0(F_2^{-1} F_3 K)$ and that stability of (E, ϕ) implies that b and c are non-zero. Denote the critical submanifold of length 3 chains $E = F_1 \oplus F_2 \oplus F_3$ with $\deg(F_i) = \delta_i$, $i = 1, 2, 3$ by $\mathcal{N}^3(\delta_1, \delta_2, \delta_3)$. Clearly, $E_1 = F_1 \oplus F_3$ and $E_2 = F_2$, in particular $\delta_2 = d_2$. With these preliminaries we have the following description of the length 3 critical submanifolds.

Proposition 3.2. *There is an isomorphism*

$$\begin{aligned} \mathcal{N}^3(\delta_1, \delta_2, \delta_3) &\to S^{m_1} X \times S^{m_2} X \times \operatorname{Pic}^{\delta_2}(X) \\ \big(F_1 \oplus F_2 \oplus F_3, \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)\big) &\mapsto \big((c), (b), F_2\big), \end{aligned}$$

where $S^m X$ denotes the mth symmetric product of X and

$$m_1 = 2g - 2 + \delta_2 - \delta_1,$$
$$m_2 = 2g - 2 + \delta_3 - \delta_2.$$

Furthermore, the Morse index of $\mathcal{N}^3(\delta_1, \delta_2, \delta_3)$ is

$$\lambda\big(\mathcal{N}^3(\delta_1, \delta_2, \delta_3)\big) = 2g - 2 + 2\delta_1 - 2\delta_3.$$

Proof. It is clear that F_2 and the divisors (b) and (c) determine the bundles F_1, F_2 and F_3 and the sections b and c up to scalar multiplication. It is easy to check that any two Higgs bundles obtained in this way are isomorphic and hence the map of the statement of the proposition is an isomorphism.

To calculate the Morse index, one simply applies (2.5), noting that $U_2 = \mathrm{Hom}(F_1, F_3)$ and so $\deg(U) = \delta_3 - \delta_1$ and $\mathrm{rk}(U_2) = 1$. □

The Poincaré polynomial of $\mathcal{N}^3(\delta_1, \delta_2, \delta_3)$ is calculated from Macdonald's formula [17] for the Poincaré polynomial of the symmetric product of an algebraic curve to be

$$P_t(\mathcal{N}^3(\delta_1, \delta_2, \delta_3)) = (1+t)^{2g} \operatorname*{Coeff}_{x^{m_1}} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \operatorname*{Coeff}_{x^{m_2}} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}, \tag{3.2}$$

where m_1 and m_2 were defined above.

This result is, however, not sufficient: for each value of $d_2 = \delta_2$ in the range (2.3) we also need to determine the possible values of the invariants δ_1 and δ_3 (or, equivalently, the invariants m_1 and m_2). To do this, note first that since m_1 and m_2 are degrees of line bundles with non-zero sections, we must have

$$m_1 \geqslant 0, \tag{3.3}$$
$$m_2 \geqslant 0. \tag{3.4}$$

However, $m_1 - m_2 = 2\delta_2 - \delta_1 - \delta_3 = 3d_2 - d$ which is strictly positive by (2.3). Hence (3.4) implies (3.3). Secondly, we get from stability applied to the bundles $F_2 \oplus F_3$ and F_3 that

$$\delta_2 + \delta_3 < \frac{2}{3}d, \tag{3.5}$$
$$\delta_3 < \frac{1}{3}d. \tag{3.6}$$

In this case (2.3) shows that (3.5) implies (3.6). The former inequality is equivalent to

$$m_2 < 2g - 2 + \frac{2}{3}d - 2d_2. \tag{3.7}$$

Note that m_1 (and hence δ_1 and δ_3) can be recovered from m_2, d and d_2:

$$m_1 = m_2 + 3d_2 - d.$$

It follows that there is a non-empty critical submanifold

$$\mathcal{N}^3(m_2) = \mathcal{N}^3(\delta_1, \delta_2, \delta_3)\big)$$

for each m_2 satisfying (3.4) and (3.7). It remains to express the Morse index in terms of m_2; this is a simple calculation giving

$$\lambda\big(\mathcal{N}^3(m_2)\big) = 2(5g - 5 + d - 3d_2 - 2m_2). \tag{3.8}$$

We now have all the ingredients required for the calculation of the Poincaré polynomial of $\mathcal{M}_{U(2,1)}$.

Theorem 3.3. *Suppose that* $(d, 3) = 1$. *The Poincare polynomial of the component* $\mathcal{M}_{d_1,d_2}$ *of* $\mathcal{M}_{U(2,1)}$ *is*

$$P_t(\mathcal{M}_{d_1,d_2}) = P_t(\mathcal{N}^2) + \sum_{m_2=0}^{i} t^{2(5g-5+d-3d_2-2m_2)} P_t(\mathcal{N}^3(m_2)), \tag{3.9}$$

where $d = d_1 + d_2$, $i = \left[\frac{2d}{3}\right] - 2d_2 + 2g - 2$ *and* $P_t(\mathcal{N}^2)$ *and* $P_t(\mathcal{N}^3(m_2))$ *are given by* (3.1) *and* (3.2) *respectively.*

It seems difficult to simplify further this expression and obtain a closed formula for the Poincaré polynomial. On the other hand, we can obtain explicit formulas in low genus. For example, consider $g = 2$ and $d = 1$, then the values allowed by (2.2) for d_2 are $d_2 = 0, 1$, in particular $\mathcal{M}_{U(2,1)}$ has two connected components. In the case $d_2 = 1$ (and hence $d_1 = 0$) we obtain

$$\begin{aligned} P_t(\mathcal{M}_{0,1}) = {} & t^{14} + 8\,t^{13} + 30\,t^{12} + 68\,t^{11} + 105\,t^{10} + 124\,t^9 \\ & + 128\,t^8 + 128\,t^7 + 127\,t^6 + 120\,t^5 + 99\,t^4 + 64\,t^3 + 29\,t^2 + 8\,t + 1 \end{aligned} \tag{3.10}$$

(these calculations were performed using the computer algebra system Maple). In the case when $d = 1$ and $d_2 = 0$ (and so $d_1 = 1$) the condition $3d_2 - d > 0$ is not satisfied, however, as noted above, the moduli space is isomorphic to the moduli space for $d = 2$ and $d_2 = 1$, which does satisfy $3d_2 - d > 0$. In this case one obtains

$$\begin{aligned} P_t(\mathcal{M}_{1,0}) = {} & 3\,t^{14} + 28\,t^{13} + 115\,t^{12} + 292\,t^{11} + 528\,t^{10} + 728\,t^9 \\ & + 795\,t^8 + 704\,t^7 + 511\,t^6 + 308\,t^5 + 161\,t^4 + 76\,t^3 + 30\,t^2 + 8\,t + 1. \end{aligned} \tag{3.11}$$

Note in particular that this shows that the two components are not homeomorphic.

4. Fixed determinant bundles and Euler characteristic

Consider the determinant map $\mathcal{M}_{d_1,d_2} \to \mathrm{Pic}^d(X)$ given by

$$(E, \phi) \mapsto \det(E).$$

Its fibre over a degree d line bundle Λ is naturally isomorphic to the moduli space of Higgs bundles of the form (2.1) with fixed determinant

Λ and $\deg(E_i)=d_i$. We denote this space by $\widetilde{\mathcal{M}}_{d_1,d_2}$; it is homeomorphic to the moduli space of reductive representations of Γ in $SU(2,1)$ with the given invariants d_1 and d_2. The moduli space $\mathcal{M}_{SU(2,1)}$ is the union of the spaces $\widetilde{\mathcal{M}}_{d_1,d_2}$ over the values of d_1 and d_2 such that (1.1) is satisfied.

The calculation of the Poincaré polynomial of $\widetilde{\mathcal{M}}_{d_1,d_2}$ proceeds in the same manner as the calculation for $\mathcal{M}_{d_1,d_2}$; the main difference is that the critical submanifolds $\widetilde{\mathcal{N}}^2$ and $\widetilde{\mathcal{N}}^3(m_2)$ now become pull-backs of 3^{2g}-fold coverings of the Jacobian in the same way as in Propositions 2.5 and 3.10 of [11], where the relevant Poincaré polynomials were also calculated. The calculation of the Morse indices is identical to the non-fixed determinant case. We omit the details and only state the result.

Theorem 4.1. *Suppose that $(d,3)=1$. The Poincare polynomial of the component $\widetilde{\mathcal{M}}_{d_1,d_2}$ of $\mathcal{M}_{SU(2,1)}$ is*

$$P_t(\widetilde{\mathcal{M}}_{d_1,d_2}) = P_t(\widetilde{\mathcal{N}}^2) + \sum_{m_2=0}^{i} t^{2(5g-5+d-3d_2-2m_2)} P_t(\widetilde{\mathcal{N}}^3(m_2)), \tag{4.1}$$

where $d=d_1+d_2$ and $i=\left[\frac{2d}{3}\right]-2d_2+2g-2$. The Poincaré polynomials $P_t(\widetilde{\mathcal{N}}^2)$ and $P_t(\widetilde{\mathcal{N}}^3(m_2))$ are given by

$$\begin{aligned} P_t(\widetilde{\mathcal{N}}^2) = {} & \frac{(1+t)^{2g}}{1-t^2} \\ & \cdot \operatorname*{Coeff}_{x^i}\left(\frac{t^{10g-10+2d-6d_2-4i}}{xt^4-1} - \frac{t^{2i+2}}{x-t^2}\right)\left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}\right), \end{aligned} \tag{4.2}$$

where $i=\left[\frac{2d}{3}\right]-2d_2+2g-2$, and

$$\begin{aligned} P_t(\widetilde{\mathcal{N}}^3(m_2)) = {} & \operatorname*{Coeff}_{x^{m_1}} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \operatorname*{Coeff}_{x^{m_2}} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \\ & + \binom{2g-2}{m_1}\binom{2g-2}{m_2}(3^{2g}-1)t^{m_1+m_2}, \end{aligned} \tag{4.3}$$

where $m_1=m_2+3d_2-d$.

It is interesting to note that the Poincaré polynomial of $\mathcal{M}_{d_1,d_2}$ is not simply the product of those of the Jacobian and $\widetilde{\mathcal{M}}_{d_1,d_2}$, in contrast to the situation for moduli spaces of stable bundles. Tensoring with a degree zero line bundle gives an action of the Jacobian of X on $\mathcal{M}_{U(2,1)}$. Furthermore, the determinant map is equivariant if we let the Jacobian act on $\operatorname{Pic}^d(X)$ by tensoring with the third power of a line bundle and hence

$$\mathcal{M}_{d_1,d_2} \cong \big(\widetilde{\mathcal{M}}_{d_1,d_2} \times \operatorname{Pic}^d(X)\big)/T_3,$$

where $T_3 \cong (\mathbb{Z}/3)^{2g}$ is the subgroup of 3-torsion points of the Jacobian. So far this is completely analogous to the case of moduli of stable vector bundles; see e.g. §9 of Atiyah and Bott [1]. In that case, the finite covering group acts trivially on the rational cohomology of the fixed determinant moduli space (this result was first proved by Harder and Narasimhan [12]) implying that the Poincaré polynomial of the non-fixed determinant moduli space is simply the product of that of the fixed determinant moduli space by that of the Jacobian. Hence our calculations imply the following result.

Proposition 4.2. *The action of T_3 on the rational cohomology of $\widetilde{\mathcal{M}}_{d_1,d_2}$ is non-trivial.*

This phenomenon also occurs for moduli of representations of Γ in $SL(3,\mathbb{C})$ (see [11]). From the point of view of the Morse theory computation the reason for this result is as follows. The critical submanifolds in the non-fixed determinant case fibre over the Jacobian via the determinant map and the fibres are isomorphic to the fixed determinant critical submanifolds. One sees from [11] that T_3 acts trivially on the the rational cohomology of the length 2 critical submanifolds while, on the other hand, the Poincaré polynomial of the length 3 submanifolds is not the product of those of the fixed determinant critical submanifold by that of the Jacobian. Thus the explanation for Proposition 4.2 from this point of view is that the action of T_3 is non-trivial on the rational cohomology of the fixed determinant length 3 submanifolds.

Another point worthy of note is that we can determine the Euler characteristic of $\widetilde{\mathcal{M}}_{d_1,d_2}$: from (2.4) and the fact that the Morse indices are even it follows that the Euler characteristic of $\widetilde{\mathcal{M}}_{d_1,d_2}$ is simply the sum of the Euler characteristics of the critical submanifolds. The formula (4.2) shows that the critical submanifold $\widetilde{\mathcal{N}}^2$ has Euler characteristic zero and, since

$$\operatorname*{Coeff}_{x^{m_i}} \frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} = (-1)^{m_i}\binom{2g-2}{m_i},$$

(4.3) shows that

$$\begin{aligned}\chi(\widetilde{\mathcal{N}}^3(m_2)) &= P_t(\widetilde{\mathcal{N}}^3(m_2))_{|t=-1} \\ &= \binom{2g-2}{m_1}\binom{2g-2}{m_2}3^{2g}(-1)^{m_1+m_2}.\end{aligned}$$

(This can also be seen directly from the fact that $\widetilde{\mathcal{N}}^3(m_2)$ is a 3^{2g}-fold covering of $S^{m_1}X \times S^{m_2}X$.) Noting that $(-1)^{m_1+m_2} = (-1)^{d+d_2}$ we therefore get

$$\chi(\widetilde{\mathcal{M}}_{d_1,d_2}) = 3^{2g}(-1)^{d+d_2}\sum_{m_2=0}^{i}\binom{2g-2}{m_2+3d_2-d}\binom{2g-2}{m_2}. \tag{4.4}$$

Thus, for example, in the case $g=2$, $d=1$, we get

$$\chi(\widetilde{\mathcal{M}}_{0,1}) = 81, \quad \chi(\widetilde{\mathcal{M}}_{1,0}) = -324.$$

In general, we see that $\widetilde{\mathcal{M}}_{d_1,d_2}$ has non-zero Euler characteristic. This also happens for representations in $SL(3,\mathbb{C})$ (see [11]) while it contrasts with the case of the moduli space of stable bundles with fixed determinant which has zero Euler characteristic, as one easily sees from the results of Desale and Ramanan [6].

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